Continuous functions

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Some figures comes from Dr. CHAN Kai Leung, thanks a lot!!

The Chinese University of Hong Kong

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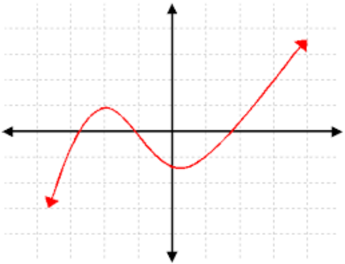
A tip first

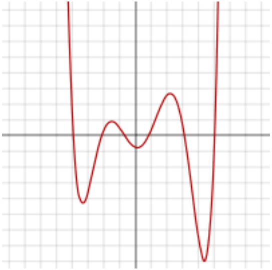
Four key words in our course:

- Limits
- Continuous function
- Differentiation
- Integration

What is a continuous function?

Inform language: A function is continuous if its graph is a single unbroken curve.





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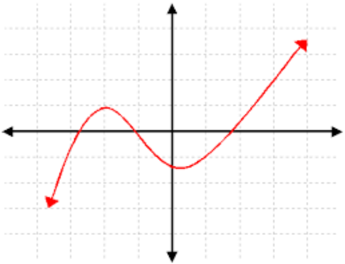
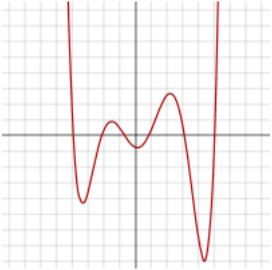
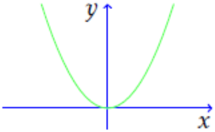


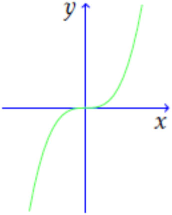


"Continuous = Connected"

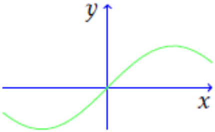
Some continuous elementary functions



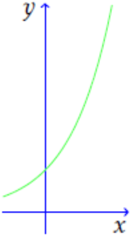
(a) $f(x) = x^2$



(b) $f(x) = x^3$



(c) $f(x) = \sin x$



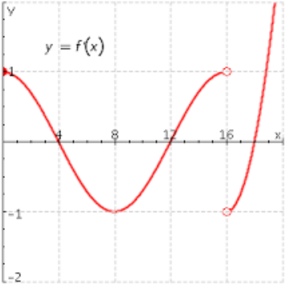
(d) $f(x) = e^x$

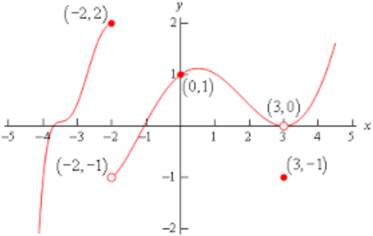
Please draw some continuous functions

Figure: Thanks for your nice pictures!!

Discontinuous functions

If f is not continuous then we call f a discontinuous function. It's graph is a broken curve.





Continuous function

Definition

Let $c \in \mathcal{D} \subseteq \mathbb{R}$ and let $f : \mathcal{D} \to \mathbb{R}$ be a function. We say that function $f(x)$ is continuous at the point c if

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Note: The set $\mathcal{D}$ will be one of the following sets:

$$\mathbb{R}, (a, b), (t, +\infty), (-\infty, s)$$

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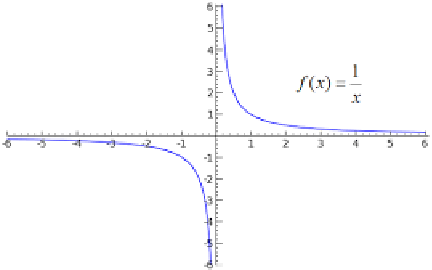
Moreover, function f is continuous at point x if and only if

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c).$$

Continuous or not, let's see



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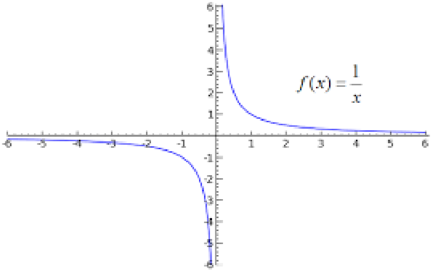

Figure: f is discontinuous on $\mathbb{R}$, but it is continuous on the interval $(0, +\infty)$ and also continuous on the interval $(-\infty, 0)$.

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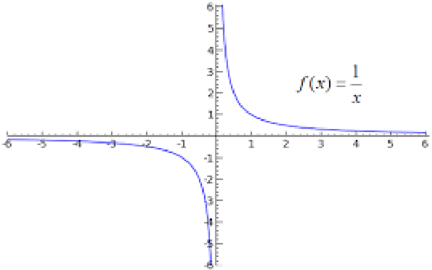

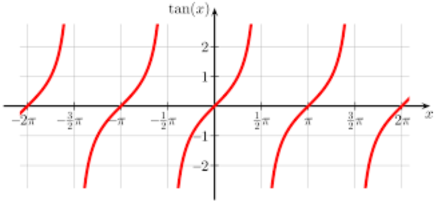

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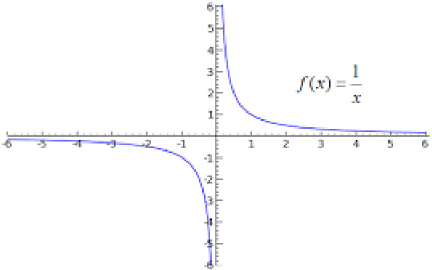

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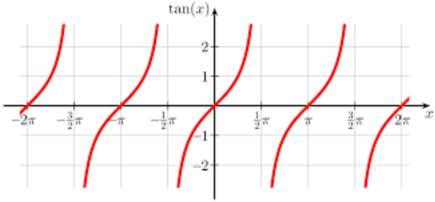

Figure: Discontinuous on $\mathbb{R}$, but it is continuous on the interval $(-\pi/2, \pi/2)$, and in fact for $k \in \mathbb{N}$ it is continuous on interval $(-\pi/2 + k\pi, \pi/2 + k\pi)$.

Example I

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Thus f is continuous at the point c. Since this holds for any $c \in \mathbb{R}$, we conclude that the function f is continuous on $\mathbb{R}$.

Example II

Is the following function f continuous at $x = 1$?

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \geqslant 1 \\ 1 - x & \text{if } x < 1 \end{cases}.$$

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Example III

Is the following function f continuous at point $x = 0$?

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \end{cases}.$$

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Yes. This follows by the fact that

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and $f(0) = 0$. This gives

$$\lim_{x \to 0} f(x) = f(0).$$

One property of continuous function

Theorem

A function f is continuous at $x = c$ if and only if for any sequence (a_n) with $a_n \neq c, \forall n \in \mathbb{N}$ and $\lim_{n\to+\infty} a_n = c$, we have

$$\lim_{n\to+\infty} f(a_n) = f\left(\lim_{n\to+\infty} a_n\right) = f(c).$$

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Let $f : [a, b] \to \mathbb{R}$. We say that f is continuous at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$. Similarly, we say that f is continuous at $x = b$ if $\lim_{x \to b^-} f(x) = f(b)$.

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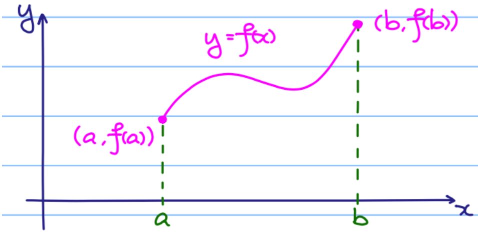

(We cannot talk about $\lim_{x \to a^-} f(x)$ and $\lim_{x \to b^+} f(x)$!)

Intermediate value theorem

Theorem

Let $f : [a, b] \to \mathbb{R}$ *be a continuous function. Suppose that* $f(a) < f(b)$. *Then for any* $f(a) < L < f(b)$ *there exists (at least one)* $c \in [a, b]$ *such that* $f(c) = L$.

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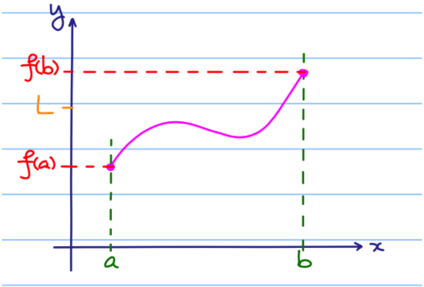

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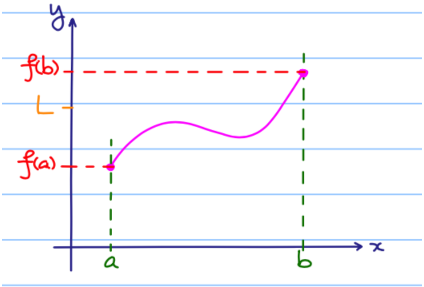
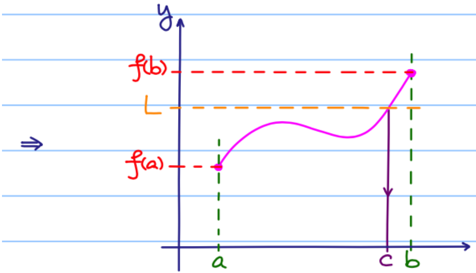


$\Rightarrow$



Maximal-Minimal theorem

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Let $f : [a, b] \to \mathbb{R}$ *be a continuous function. Then there exist* $x_m, x_M \in [a, b]$ *such that*

$$f(x_m) \leqslant f(x) \leqslant f(x_M), \quad \forall x \in [a, b].$$

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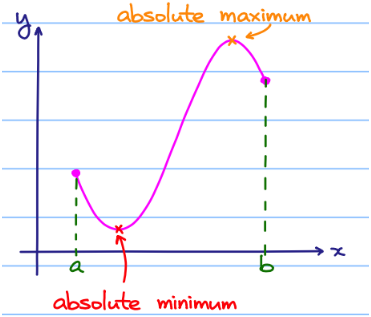
Corollary: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exits $Q > 0$ such that

$$-Q \leqslant f(x) \leqslant Q.$$

Or we say that f is a bounded function.

Maximal-Minimal theorem



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